12.5: 10;

10. $m = \int_0^{\pi/2}\int_0^{\cos x} x\,dy\,dx = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1,$

$M_y = \int_0^{\pi/2}\int_0^{\cos x} x^2\,dy\,dx = \int_0^{\pi/2} x^2\cos x\,dx = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \frac{\pi^2}{4} - 2$, and

$M_x = \int_0^{\pi/2}\int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}x\cos^2 x\,dx = \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}.$

Hence $m = \dfrac{\pi - 2}{2}$, $(\overline{x}, \overline{y}) = \left(\dfrac{\pi^2 - 8}{2(\pi - 2)}, \dfrac{\pi + 2}{16}\right)$.

12.6: 8, 10, 24, 28

8. $\mathbf{r}_u = \langle \cos v, \sin v, 0\rangle$, $\mathbf{r}_v = \langle -u\sin v, u\cos v, 1\rangle$, and $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$. Then

$$A(S) = \int_0^{\pi}\int_0^1 \sqrt{1+u^2}\,du\,dv = \int_0^{\pi} dv\ \int_0^1 \sqrt{1+u^2}\,du$$
$$= \pi\left[\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln\left|u + \sqrt{u^2+1}\,\right|\right]_0^1 = \frac{\pi}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$

10. A parametric representation of the surface is $x = y^2 + z^2$, $y = y$, $z = z$ with $0 \le y^2 + z^2 \le 9$.
Hence $\mathbf{r}_y \times \mathbf{r}_z = (2y\,\mathbf{i} + \mathbf{j}) \times (2z\,\mathbf{i} + \mathbf{k}) = \mathbf{i} - 2y\,\mathbf{j} - 2z\,\mathbf{k}$.

Note: In general, if $x = f(y,z)$ then $\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} - \dfrac{\partial f}{\partial y}\mathbf{j} - \dfrac{\partial f}{\partial z}\mathbf{k}$, and $A(S) = \iint_D \sqrt{1 + \left(\dfrac{\partial f}{\partial y}\right)^2 + \left(\dfrac{\partial f}{\partial z}\right)^2}\,dA$. Then

$$A(S) = \iint_{0 \le y^2+z^2 \le 9} \sqrt{1 + 4y^2 + 4z^2}\,dA = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} d\theta\ \int_0^3 r\sqrt{1+4r^2}\,dr = 2\pi\left[\frac{1}{12}(1+4r^2)^{3/2}\right]_0^3 = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$

24. We first find the area of the face of the surface that intersects the positive y-axis. A parametric representation of the surface is $x = x$, $y = \sqrt{1 - z^2}$, $z = z$ with $x^2 + z^2 \le 1$. Then $\mathbf{r}(x, z) = \langle x, \sqrt{1 - z^2}, z \rangle \quad \Rightarrow \quad \mathbf{r}_x = \langle 1, 0, 0 \rangle$, $\mathbf{r}_z = \langle 0, -z/\sqrt{1 - z^2}, 1 \rangle$ and $\mathbf{r}_x \times \mathbf{r}_z = \langle 0, -1, -z/\sqrt{1 - z^2} \rangle \quad \Rightarrow \quad |\,\mathbf{r}_x \times \mathbf{r}_z\,| = \sqrt{1 + \frac{z^2}{1 - z^2}} = \frac{1}{\sqrt{1 - z^2}}$.

$$A(S) = \iint\limits_{x^2 + z^2 \le 1} |\,\mathbf{r}_x \times \mathbf{r}_z\,|\, dA = \int_{-1}^{1} \int_{-\sqrt{1 - z^2}}^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz$$

$$= 4 \int_0^1 \int_0^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz \quad \text{[by the symmetry of the surface]}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t \to 1^-} 4 \int_0^t \int_0^{\sqrt{1 - z^2}} \frac{1}{\sqrt{1 - z^2}}\, dx\, dz = \lim_{t \to 1^-} 4 \int_0^t \frac{\sqrt{1 - z^2}}{\sqrt{1 - z^2}}\, dz = \lim_{t \to 1^-} 4 \int_0^t dz = \lim_{t \to 1^-} 4t = 4$$

Since the complete surface consists of four congruent faces, the total surface area is $4\,(4) = 16$.

Alternate solution: The face of the surface that intersects the positive y-axis can also be parametrized as $\mathbf{r}(x, \theta) = \langle x, \cos\theta, \sin\theta \rangle$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $x^2 + z^2 \le 1 \quad \Leftrightarrow \quad x^2 + \sin^2\theta \le 1 \quad \Leftrightarrow$ $-\sqrt{1 - \sin^2\theta} \le x \le \sqrt{1 - \sin^2\theta} \quad \Leftrightarrow \quad -\cos\theta \le x \le \cos\theta$. Then $\mathbf{r}_x = \langle 1, 0, 0 \rangle$, $\mathbf{r}_\theta = \langle 0, -\sin\theta, \cos\theta \rangle$ and $\mathbf{r}_x \times \mathbf{r}_\theta = \langle 0, -\cos\theta, -\sin\theta \rangle \quad \Rightarrow \quad |\,\mathbf{r}_x \times \mathbf{r}_\theta\,| = 1$, so $A(S) = \int_{-\pi/2}^{\pi/2} \int_{-\cos\theta}^{\cos\theta} 1\, dx\, d\theta = \int_{-\pi/2}^{\pi/2} 2\cos\theta\, d\theta = 2\sin\theta \big]_{-\pi/2}^{\pi/2} = 4$. Again, the area of the complete surface is $4(4) = 16$.

28. $x = b\cos\theta + a\cos\alpha\cos\theta$, $y = b\sin\theta + a\cos\alpha\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\cos\theta, -a\sin\alpha\sin\theta, a\cos\alpha \rangle$, $\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0 \rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\cos\theta - a^2\cos\alpha\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\cos\theta - a^2\sin\alpha\cos^2\theta\right)\mathbf{j} \\ &\quad + \left(-ab\cos^2\alpha\sin\theta - a^2\cos^2\alpha\sin\theta\cos\theta - ab\sin^2\alpha\sin\theta - a^2\sin^2\alpha\sin\theta\cos\theta\right)\mathbf{k} \\ &= -a(b + a\cos\alpha)[(\cos\theta\cos\alpha)\,\mathbf{i} + (\sin\theta\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\cos^2\alpha + \sin^2\theta\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.
Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence
$A(S) = \int_0^{2\pi} \int_0^{2\pi} a(b + a\cos\alpha)\, d\alpha\, d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.